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# A GENERAL METHOD OF DEDUCING THE EQUATION OF A TANGENT TO A CURVE.

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The following method of finding the equation of a tangent to a curve is more general than those usually given in texts in elementary analytical geometry, but can readily be substituted for any of them. At the same time it has many additional advantages.

Through any point $P\equiv(x', y')$ draw a line $l$ making an angle $\theta$ with $OX$. If this line has a point, or points, in common with a locus given by a rational integral equation, denote such a point by $Q$. If $PQ=r$, the coördinates of $Q$ are

$$x'+r\cos\theta,\quad y'+r\sin\theta.$$

Since $Q$ is on the locus, its coördinates satisfy the equation of the locus. Substituting, and arranging in ascending powers of $r$, we have

$$0=u_0+ru_1+r^2u_2+\text{.........}\qquad(a)$$

If $u_0=0$, one value of $r$ is zero; that is, one position of $Q$ is coincident with $P$, or, the point $P$ is on the locus.

Another value of $r$ will be zero and the line $l$ is said to have two points in common with the locus at $P$, if $\theta$ be chosen so that $u_1=0$. The line $l$ is then said to be a *tangent* at $P$. [Notice that we do not have $Q$ to coincide with $P$, and then speak of the tangent as a line through two *coincident* points; neither do we have $Q$ approach $P$, and define the tangent as the limit of the line $PQ$. We are not troubled with the idea of limits.] The equation of $l$ is then

$$y-y'=(x-x')\tan\theta,$$

where $\theta$ has the value which makes $u_1=0$. [If $\theta=90°$, the equation of $l$ is $x-x'=0$.]

Example I. Find the equation of the tangent at the point $P\equiv(x', y')$ to the locus $y^2=4ax$. If a line $l$ through $P$ making an angle $\theta$ with $OX$ have a point $Q$ in common with the locus its coördinates

$$x'+r\cos\theta,\quad y'+r\sin\theta,$$

where $PQ=r$, satisfy the equation of the locus.

Consequently $0=(y'^2-4ax')+2r(y'\sin\theta-2a\cos\theta)+r^2\sin^2\theta$.

Since $P$ is on the locus,

$$y'^2-4ax'=0.\qquad(1)$$

Hence one value of $r$ is zero for all values of $\theta$. Another value will be zero, that is, the line will be a tangent, if $y'\sin\theta-2a\cos\theta=0$. The equation of the tangent is therefore

$$(y-y')y'=(x-x')2a,$$

or, by using (1),
$$yy'=2a(x+x').$$

If $u_0=0$ and $u_1=0$ for all values of $\theta$, then two values of $r$ are zero for any position of $l$ through the point $P$, which is then called a *double point* on the locus. When this is the case, the line $l$ is called a tangent when $\theta$ is chosen so that $u_2=0$, thus making one more value of $r$ zero.

Example II. Find the tangent at the origin to the locus $x^2-y^2-3y^3=0$. If a line $l$ through the origin making an angle $\theta$ with $OX$ have a point $Q$ in common with the locus, its coördinates,

$$r\cos\theta, \quad r\sin\theta,$$

where $PQ=r$, satisfy the equation.

Thus $r^2(\cos^2\theta-\sin^2\theta)-3r^3\sin^3\theta=0$.

Two values of $r$ are zero for all values of $\theta$. Hence the origin is a double point of the locus. Another value of $r$ will be zero, that is, the line $l$ will be a tangent, if $\cos^2\theta-\sin^2\theta=0$; that is, if $\tan\theta=\pm1$. We get therefore two positions of $l$ satisfying the required condition, its equation being $y=x$ or $y=-x$.

Example III. If an equation of the second degree represents a locus with a double point, it is the equation of two straight lines. Denote the equation by

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

and assume that it has a double point $P\equiv(x', y')$. Through $P$ draw a line $l$ making an angle $\theta$ with $OX$, and denote any point common to the line and the locus by $Q$. The coördinates of $Q$, viz.,

$$x'+r\cos\theta, \quad y'+r\sin\theta,$$

where $PQ=r$, satisfiy the equation of the locus.

Hence $0=ax'^2+2hx'y'+by'^2+2gx'+2fy'+c+2r[(ax'+hy'+g)\cos\theta$
$+(hx'+by'+f)\sin\theta]+r^2[a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta]$.

Since at a double point two values of $r$ must be zero, for all values of $\theta$, we must have

$$ax'^2+2hx'y'+by'^2+2gx'+2fy'+c=0, \tag{1}$$

$$ax'+hy'+g=0, \tag{2}$$

$$hx'+by'+f=0. \tag{3}$$

Multiplying (2) by $x'$, (3) by $y'$, and subtracting their sum from (1), we have

$$gx'+fy'+c=0. \tag{4}$$

Eliminating $x'$, $y'$ from (2), (3), (4), we have

$$\begin{vmatrix} a, & h, & g, \\ h, & b, & f, \\ g, & f, & c, \end{vmatrix} =0,$$

which is the condition that the given equation represents two straight lines.

EXAMPLE IV. Find the tangent at the origin to the locus $x^2+2y^2+3y^3=0$. Proceeding as in example II, we get

$$r^2(\cos^2\theta+2\sin^2\theta)+3r^3\sin^3\theta=0.$$

Two values of $r$ are zero, so that the origin is a double point on the locus. No value of $\theta$ can be found which will make another value of $r$ zero. Hence the locus has no real tangent at the origin. Such a point is called a *conjugate* point, and if we were to trace the curve we would find that we could not find another point satisfying the equation and as near as we pleased to the origin. [While it is true that when there are no tangents at a point on a locus, it is a conjugate point, it does not hold conversely that at a conjugate point tangents are not obtained by the usual methods. For example, we obtain $y=0$ as the tangent at the origin to the locus $y^2=2x^2y+x^4y-2x^4$, although the origin is a conjugate point.]

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If $u_0=0$, and the value of $\theta$ which makes $u_1=0$ also makes $u_2=0$, but does not make $u_3=0$, the equation gives three zero values of $r$ instead of two for the position of $l$ given by $u_1=0$. Such a point is called a *point of inflexion*.

EXAMPLE V. Find the tangent at the origin to the locus $x^3+x+y=0$. Proceeding as in example II we get

$$0=r(\cos\theta+\sin\theta)+r^3\sin^3\theta.$$

If $\theta$ be chosen so that $\cos\theta+\sin\theta=0$, we get *two* additional zero values for $r$. Hence the origin is a point of inflexion, the tangent at that point being $x+y=0$.